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Note

Note on a Result of Erdös

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Forty years ago, Erdös [1] made the following observation:
If $P(x) \equiv c_0 + c_1 x + \cdots + c_n x^n$, $n \geqq 0$, $c_0 c_n \neq 0$, *has integral coefficients only and has all its zeros in* $(-1, 1)$, *then*

$$|c_n| \geqq 2^{n/2}.$$

We prove here the following

THEOREM. *If* $P(x) \equiv c_n(x-x_1)(x-x_2)\cdots(x-x_n)$ $(n \geqq 1)$ *has all its zeros in* $[-1, 1]$ *and satisfies, for* $k = 0, 1, ..., j(\geqq 1)$,

$$|P(\eta_k)| \geqq a > 0, \qquad \eta_k = \cos(k\pi/j),$$

then

$$|c_n| \geqq 2^{(j-1)n/j} a.$$

Equality holds for $P(x) \equiv \pm a T_j^{n/j}(x)$ $(a > 0)$ *whenever j divides n, where* T_j *is the jth degree Chebyshev polynomial of the first kind.*

Remark. If $a = 1, j = 2$, the result of Erdös follows.

Proof.

$$a^{2j} \leqq \left| P(\eta_0)\, P(\eta_j) \prod_{k=1}^{j-1} P^2(\eta_k) \right|$$

$$= |c_n|^{2j} \prod_{k=1}^{n} (j2^{j-1})^{-2} (1 - x_k^2)(T_j'^2(x_k)$$

(an "empty" product means 1).

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Set

$$x_k = \cos \theta_k, \qquad 0 \leqq \theta_k \leqq \pi, \; k = 1, 2, ..., n.$$

Then

$$(1 - x_k^2)\, T_j'^2(x_k) = j^2 \sin^2 j\theta_k \leqq j^2.$$

Hence,

$$a^{2j} \leqq |c_n|^{2j}\, 2^{-2(j-1)n}.$$

Reference

1. Paul Erdös, Some remarks on polynomials, *Bull. Amer. Math. Soc.* (*N.S.*) **53** (1947), 1169–1176.